Equilibrium Selection in Global Games with Strategic Substitutes

by

Rodrigo Harrison*

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Abstract

This paper proves an equilibrium selection result for a class of games with strategic substitutes. Specifically, for a general class of binary action, N-player games, we prove that each such game has a unique equilibrium strategy profile. Using a global game approach first introduced by Carlsson and van Damme (1993), recent selection results apply to games with strategic complementarities. The present paper uses the same approach but removes the assumption of perfect symmetry in the *dominance region* of the players' payoffs. Instead we assume that players are ordered such that asymmetric dominance regions overlapped sequentially. This allow us to extend selection results to a class of games with strategic substitutes.

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*Department of Economics, Georgetown University U.S.A. and Departamento de Industrias UTFSM Chile. Email: harrisor@georgetown.edu. I am specially grateful to Roger Lagunoff for his advice and help. I thank Luca Anderlini and Axel Anderson for valuable discussion and Stephen Morris for helpful comments on an earlier version of this paper. I also thank participants at SETI 2001 and Midwest Economic Theory 2002 conferences.

1 Introduction

In general, game-theoretic models are developed under the assumption that the rational behavior of the players and the structure of the game are common knowledge. Since these assumptions might be too stringent for modeling real-life players, it is important to know whether the prediction of a game substantially changes in comparison to the predictions of a slightly altered version of the same game[1]. If indeed it turns out that only certain of the game's equilibria survive this "robustness check," then we may reasonably refine our prediction of what happens in such games.

This paper examines the dual issues of equilibrium selection and robustness in a class of games with strategic substitutes. These are games in which each player's marginal payoff from increasing his own action is decreasing in the other players' actions. The standard example is the game of voluntary contribution toward a public good. The equilibria exhibit a classical free rider problem: an individual is less willing to contribute the larger is the total contribution of others. If one's contribution is an indivisible choice such as a unit of time or effort, then voluntary contribution games typically exhibit multiple Nash equilibria, each corresponding to a distinct configuration of contributors and non-contributors.

To examine equilibrium selection in games such as these, we follow the *Global Games* approach pioneered by Carlsson and van Damme (1993).[2] The idea in this approach is to examine Nash equilibria as a limit of equilibria of payoff-perturbed games. More formally, suppose G is a standard game of complete information where the payoffs depend on a parameter $x \in \mathbb{R}$, and also suppose that for some subset of the parameter x, G has a strict Nash equilibrium. Rather than observing the parameter x, suppose instead that each player observes a private signal $x_i = x + \sigma\varepsilon_i$ where

[1] Examples in this direction are the seminal contributions of Harsanyi's games with randomly disturbed payoffs (Harsanyi (1973)), and Selten' concept of trembling hand perfection (Selten (1975)).

[2] For an excellent description and survey of the ensuing literature see Morris and Shin (2000).

$\sigma > 0$ is a scale factor and ε_i is a random variable with density ϕ. Denote this "perturbed game" by $G(\sigma)$, and let $NE(G)$ and $BNE(G(\sigma))$ denote the sets of Nash and Bayesian Nash equilibria of the unperturbed and perturbed games, respectively. Equilibrium selection is obtained when $lim_{\sigma \to 0} BNE(G(\sigma))$ is a subset of $NE(G)$.

Carlsson and van Damme (1993) show, in fact, that for two-player, two-action games, this limit comprises a single equilibrium profile. Moreover, this equilibrium profile is obtained through iterated deletion of strictly dominated strategies. Roughly, the deletion requires that, for each player and for each action of that player, there are certain extreme values of the parameter, x, for which that action is strictly dominant. Even if these values carry very little probability weight, the players can use signals close to these "dominance regions" to rule out certain types of behavior of others. Hence, the iterative deletion proceeds.

Recently, these results have been extended by Frankel, Morris and Pauzner (2002) for games with many players and many actions. However, existing results in this literature are tipically limited to the case of strategic complementarities (and some other technical assumptions). This strong result is very useful for many games such as bank run models (Goldstein and Pauzner (2000)), currency crises games (Morris and Shin (1998)), etc.

Yet, there is a wide class of games where this condition is not satisfied. The voluntary provision example mentioned above is but one example. Of course in the two player case, the game can be represented as a game of strategic complements by just reordering the set of actions. However, in games with more than two players , the analysis has not been extended to games of strategic substitutes.

The key insight in the present paper is to show how global games ideas can apply to certain games of strategic substitutes when the players' payoffs display a certain, commonly known asymmetry. Specifically for a class of binary actions games, we assume that there exists an ordering of players such that each player's dominance region is an arbitrary displacement to the right of the "previous" player's dominance region, i.e. the values of x at which some player's upper dominance region begins and at which his lower dominance region finishes are strictly higher (lower) compared to those of any lower (higher) player. Under these assumptions and some other technical properties, the main result of the paper proves that there exists a unique equilibrium profile. Specifically, we show that as the noise goes to zero, a process of iterated elimination of *conditionally* dominated strategies converges to a single profile of *switching strategies*. In such a profile, each player has a threshold, cutoff signal above which he takes the higher (contributing) action, and below which the lower (non-contributing) action is taken. A very important characteristic of this profile is that each player has a different cutoff point. Interestingly, the order of these cutoff points is the same order that the players have. That is, the lower the player in the ordering, the smaller is his threshold. More precisely, the equilibrium predicts that the first player switches at the end of his lower dominance region, the last player switches at the beginning of his upper dominance region and all the other players have switching points in between these two. Intuitively, the equilibrium selected establishes that, if there are certain number of players choosing the contributing action, it must be the case that they are the lowest according to the players' order, conditional on the value of the parameter. Therefore, depending on the specific payoff structure of the game, the equilibrium profile structure might play an interesting role from an efficiency point of view. The result suggests that common knowledge of the order of players and global games structure are sufficient conditions to select not only a unique but also an ordered equilibrium.

As an introductory example, in Section 2, we present a game of public good provision, where all the assumptions are satisfied. The main result for this game is that for general distributional properties of the signal noise, there exists a unique strategy profile played in equilibrium. This profile induces an efficient provision of the public good, and the contributions come from the lowest cost contributors. This result suggests that inefficient contribution equilibria survive only under a pair of stringent assumptions: common knowledge of the fundamentals, and perfect symmetry in the players' characteristics. In section 3 we present this general framework and establish our main result. In sections 4 we develop the main steps of the proof and finally, in section 5 we presents the conclusions. Proofs of propositions and lemmas are relegated to the appendix.

2 Example: Public Good Provision

In many collective action problems multiple Nash equilibria may exist, each corresponding to a different configuration of contributors. Many of these equilibria are inefficient since individuals with a higher marginal cost of contributing end up contributing disproportionately. Here, we prove a result that suggests that these inefficient Nash equilibria are not robust.

We develop a binary action game of incomplete information in which the mechanism for public good provision utilizes both government and voluntary contributions. In particular, to fund a public good, a government pledges "seed money" which must be augmented by funds from private contributors. Each contributor, upon receiving a private signal of the amount of this pledge, then chooses whether to contribute. Agents have costs of contributing.

2.1 The Game

Consider the following N person game M_N. A government (or a social planner) decides to provide a public good G, requiring society's contribution. The society is composed by N different individuals indexed by $i = 1, 2..., N$. Each agent has to decide whether to contribute, choosing an indivisible action a_i from the binary set

$A_i = \{1 = \text{contribute}, 0 = \text{not contribute}\}$.

Let $G(x, n)$ denote the public good technology, where $x \in [\underline{X}, \overline{X}]$ is the government contribution and n is the number of people who decide to contribute (not considering the player i). Without loss of generality we can characterize the payoffs as follows: if the agent i chooses to contribute, he has to provide an effort (contribution) $c_i > 0$, and receives a utility $G(x, n + 1) - c_i$. On the other hand, if the same agent chooses not to contribute (free ride), he will receive a utility $G(x, n)$. Let be $\Delta G(x, n) = G(x, n+1) - c_i - G(x, n)$ the player i'net payoff from contributing. Finally, the assumptions about the mechanism are the following:

(a.1) Strategic Substitutes. The greater the number of people contributing the smaller is player i's incentive to contribute.

$\Delta G(x, n) < \Delta G(x, n - 1)$

Where $G(x, n = 0) \geq 0$.

(a.2) Continuity and Differentiability

$\forall n$ $G(x, n)$ is a continuous and differentiable function of x.

(a.3) Monotonicity

$\Delta G(x, n)$ is an increasing function of x. i.e. $\frac{\partial \Delta G(x,n)}{\partial x} > 0$ $\forall x \in [\underline{X}, \overline{X}]$.

(a.4) Dominance Regions. Conditional on the value of the government contribution: $\exists$ $\overline{k}_i < \overline{X}$ solving $\Delta G(x, N - 1) = 0$, i.e. $\forall x > \overline{k}_i$ action 1 is a strictly dominant strategy, and $\exists$ $\underline{k}_i > \underline{X}$ solving $\Delta G(x, 0) = 0$, i.e. $\forall x < \overline{k}_i$ action 0 is a strictly dominant strategy.

Assumption **(a.1)** states the condition in the payoff structure such that this game is a game of strategic substitutes. In general, the greater the other players' strategy profile, the smaller is player i's incentive to increase his strategy. Assumption **(a.2)** establishes a continuity and differentiability condition in the government contribution variable (the exogenous parameter), while **(a.3)** establishes that the higher the government's contribution, the greater the player's incentive to contribute. Finally, assumption **(a.4)** requires that for a sufficiently high (low) values of the government contribution, player i will always (never) contribute, i.e. (not) contributing is a strictly dominant strategy.

2.2 Incomplete Information

Suppose now that the game is characterize by incomplete information in the payoff structure. Instead of observing the actual value of the government contribution x, each player just observes a private signal x_i, which contains diffuse information about x. The signal has the following structure: $x_i = x + \sigma\varepsilon_i$, where $\sigma > 0$ is a scale factor, x is drawn from $\left[\underline{X}, \overline{X}\right]$ with uniform density and ε_i is an independent realization of the density ϕ with support in $[-\frac{1}{2}, \frac{1}{2}]$. We assume ε_i is $i.i.d.$ across the individuals.

In this context of incomplete information, a Bayesian pure strategy for player i is a function $s_i : [\underline{X} - \frac{1}{2}\sigma, \overline{X} + \frac{1}{2}\sigma] \to A_i$, and S_i is the set that contain all such strategies. A pure strategy profile is a vector $s = (s_1, s_{2,}...s_N)$, where $s_i \in S_i$ for all i and equivalently define $s_{-i} = (s_1, s_2, ..s_{i-1}, s_{i+1}, ...s_N) \in S_{-i}$.

Defining this game of incomplete information as $M_N(\sigma)$, let us define $BNE(M_N(\sigma))$ as the set of Bayesian Nash equilibria of $M_N(\sigma)$. For simplicity we will restrict the analysis to the two player case, but the extension to the many players case is a direct application from of our main result.

2.3 Two Player Case

We can represent the two player case in the following normal form:

		Player 2	
		$a_2 = 1$	$a_2 = 0$
Player 1	$a_1 = 1$	$G(x,2) - c_1,\quad G(x,2) - c_2$	$G(x,1) - c_1,\quad G(x,1)$
	$a_1 = 0$	$G(x,1),\quad G(x,1) - c_2$	$G(x,0),\quad G(x,0)$

First suppose $c_1 = c_2$, the symmetric case. Then both players have the same payoff and dominance regions. In figure 1, we graphically describe the dominance region structure of the game, where the cutoff points are the same (i.e. $\underline{k}_1 = \underline{k}_2 = \underline{k}$ and $\overline{k}_1 = \overline{k}_2 = \overline{k}$). Thus, if $x > \overline{k}$ $(x < \underline{k})$ both players are in the upper (lower) dominance region.

In the case of complete information about x, the set of Nash equilibria has the following structure:

- For values of x in the dominance regions, both players choose the dominant action. In figure 1 the dashed lines denote the value of x for which player 1 is choosing a dominant action, and solid lines denote when player 2 is choosing dominant actions. Therefore in each dominance region there exists a unique action profile in equilibrium: $a = (a_1 = 1, a_2 = 1)$ in the upper region and $a = (a_1 = 0, a_2 = 0)$ in the lower region.

- If x takes values in the interval $\left(\underline{k}, \overline{k}\right)$ there are two pure strategy Nash equilibria, where one player chooses to contribute and the other chooses not to contribute. The feasible profiles played in equilibrium are either $a = (a_1 = 1, a_2 = 0)$ or $a = (a_1 = 0, a_2 = 1)$

This Nash equilibria structure suggests two important observations. First, the Carlsson and van Damme equilibrium selection result can not be applied to this game

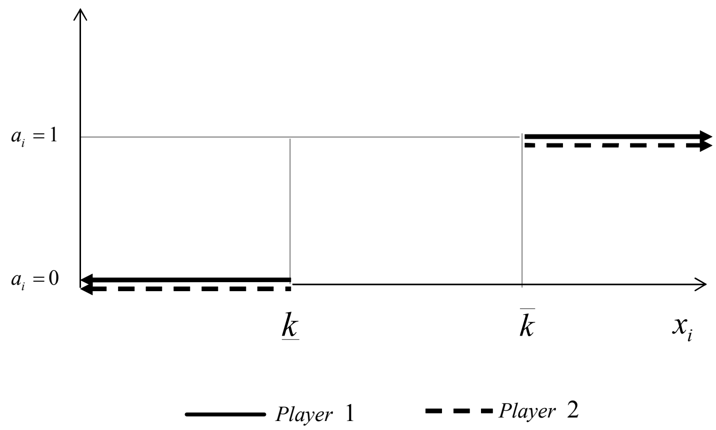

Figure 1: Symmetric Case

because it requires that a selected equilibrium be a unique Nash equilibrium for some subset of values of the exogenous parameter (x in this case). In this game, neither the strategy profile $a = (0, 1)$ nor $a = (1, 0)$ is a unique Nash equilibrium for some value of x. Second, for $x \in \left(\underline{k}, \overline{k}\right)$ this symmetric case not only implies multiplicity, but also that each of the equilibria has an asymmetric structure where just one of the players contributes. This suggests that it is likely that asymmetry will play an important role in any equilibrium selection attempt.

Let us introduce asymmetry in the payoff structure of the game. Suppose now that $c_2 > c_1$, and without loss of generality let us assume $c_1 = c$ and $c_2 = (1 + \delta)c$ where $\delta > 0$.

In figure 2 we can observe that the asymmetry generates the overlapping of the dominance regions. A very important consequence of this fact is the generation of a subset of values of x, $(\underline{k}_1, \underline{k}_2) \cup (\overline{k}_1, \overline{k}_2)$, where the profile $a = (1, 0)$ is the unique equilibrium. This enable us to apply the Carlsson and van Damme result.

Define s^* as a particular profile of switching strategies, such that player 1 and player 2 switches from action 0 to action 1 at the cutoff points $\underline{k}_1$ and $\overline{k}_2$ respectively.

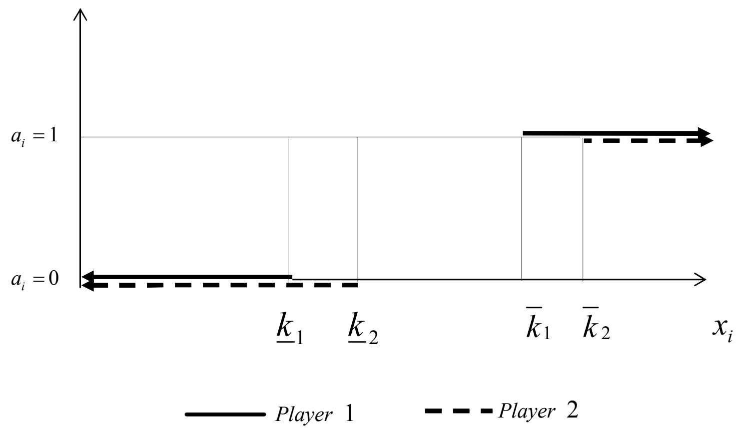

Figure 2: Asymmetric Case

We can summarize the result in the following proposition:

Proposition. *Consider a game $M_2(\sigma)$ satisfying assumptions (a1) to (a4). There exists a unique strategy profile s^* that survives iterated deletion of the strictly dominated strategies for a sufficient small amount of noise, so that $\exists\ \overline{\sigma} > 0$, s.t. $\forall \sigma \in (0,\overline{\sigma})$, $BNE(M_2(\sigma)) = \{s^*\}$.*

Figure 3 shows the structure of the equilibrium profile s^*: player 1 switches from not contributing to contributing at $\underline{k}_1$; and player 2 switches at $\overline{k}_2$. It is important to notice that this strategy profile induces an efficient provision of the public good, and that the contributions come from the lowest cost contributors. The result suggests that inefficient contribution equilibria survive only under a pair of stringent assumptions: common knowledge of the fundamentals, and perfect symmetry in the players' characteristics.

Since the existence of overlapped dominance regions allowed us to select a particular equilibrium, it suggests that generalizing this payoff structure, under the global games approach, we can prove the existence of a unique equilibrium in a more general

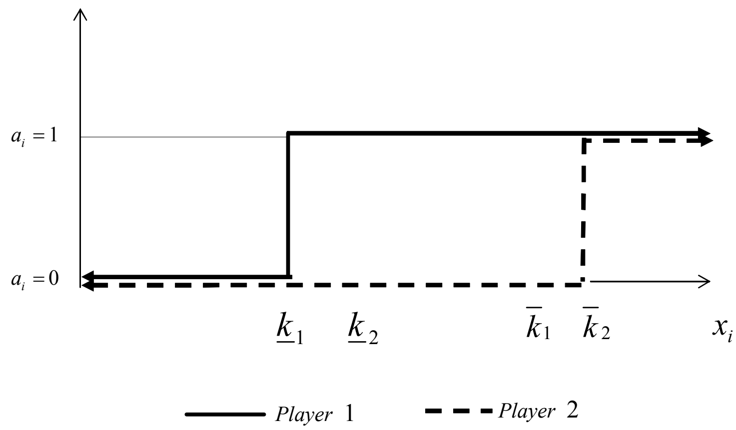

Figure 3: Equilibrium Selection: Two Players Case

class of games with strategic substitutes.

The next sections develop a more general framework, states and prove our main result: the existence of a unique equilibrium profile in certain class of global games with strategic substitutes.

3 General Framework

Consider the following general setup for an N person game G_N. There are N anonymous players indexed by i and each player has a binary set of actions $A_i = \{0, 1\}$.[3] Player i' payoff function is $\pi_i(a_i, n, x)$ where $a_i \in A_i$, $n \in \{0, ..., N-1\}$ is the number of players (other than i) that are choosing action 1 and x is an exogenous variable which takes values in the interval $[\underline{X}, \overline{X}] \subset \mathbb{R}$.

Finally let us define $\Delta\pi_i(n, x) = \pi_i(1, n, x) - \pi_i(0, n, x)$ as agent i's payoff difference when he is choosing action 1 rather than action 0. We consider the following assumptions for the payoff structure:

[3] We will also refer to $a_i = 0$ as the "lower" action and $a_i = 1$ as the "higher" action.

(A1). Strategic Substitutes (SS). Conditional on the value of x, the greater the other players' strategy profile, the smaller is player i's incentive to choose the higher action:

If $n > n'$ $\Delta\pi_i(n, x) < \Delta\pi_i(n', x)$ $\forall x$.

(A2). Continuity (C)

$\pi_i(a_i, n, x)$ is a continuous function of x.

(A3). Monotonicity (M). The greater the value of the exogenous variable x, the greater the player i's incentive to choose the higher action:

$\exists$ $c > 0$ $s.t.$ if x, $x' \in [\underline{X}, \overline{X}]$ and $x \geq x'$, then

$\Delta\pi_i(n, x) - \Delta\pi_i(n, x') \geq c(x - x')$ $\forall n$.

(A4). Upper and Lower Indifference Signals (IS). If other players are choosing identical actions, there exists a unique value of x such that player i is indifferent between the two actions:

$\forall i$ $\exists$ $\underline{k}_i > \underline{X}$ $s.t.$ $\Delta\pi_i(0, \underline{k}_i) = 0$ and $\exists$ $\overline{k}_i$ $s.t.$ $\overline{X} > \overline{k}_i > \underline{k}_i$ $s.t.$ $\Delta\pi_i(N-1, \overline{k}_i) = 0$.

(A5). Player Order (PO) Player j will be "greater" than player i, if for both players observing the same value of x and facing the same strategy profile, player j has less incentive to pick the higher action (i.e. gets a lower net payoff):

There exists a players order $\{1, ..., N\}$ such that $\exists$ $\alpha > 0$ $s.t$ if $j > i$ then $\Delta\pi_i(n, x) - \Delta\pi_j(n, x) > \alpha$ $\forall i, j$ $\forall n$.

An important remark is that assumptions A1 (SS), A3 (M) and A4 (IS) provide sufficient conditions for the existence of dominance regions, along which each action is strictly dominant. This fact provides this setup with the necessary global game structure, i.e. $\forall\, x < \underline{k}_i$ $\Delta\pi_i(n, x) < 0$ and $\forall\, x > \overline{k}_i$ $\Delta\pi_i(n, x) > 0$ $\forall n$.

Additionally, these assumptions allow us to state a more general single crossing property, which will help to characterize the equilibrium profile:

Lemma 1. *There exists a unique* $\widetilde{x} \in [\underline{X}, \overline{X}]$ *solving* $\Delta\pi_i(n, \widetilde{x}) = 0$.

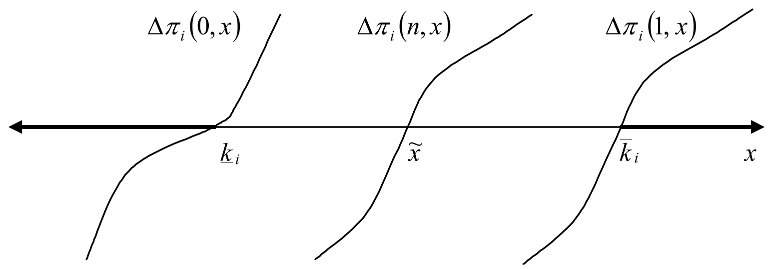

Figure 4: Player i's payoffs dependence on x

Therefore $\Delta\pi_i(n,x) < 0\ \forall\, x < \widetilde{x}$ *and* $\Delta\pi_i(n,x) > 0\ \forall\, x > \widetilde{x}\ \forall i\ \forall n.$

In figure 4, we can observe how player i's payoffs depend on x. From lemma 1 we know that for all n there exists a unique $\widetilde{x}$ such that player i is indifferent between the two actions, i.e. given n, player i's best response is to switch from the lower action to the higher action at a unique value of the signal. Given assumption A3 (M) we can also conclude that the net payoff function is monotonic in x and by assumption A1 (SS) we know that for different n the net payoff functions do not intersect each other.

Assumption A5 (PO) directly implies that if $j > i$ then $\underline{k}_j > \underline{k}_i$ and $\overline{k}_j > \overline{k}_i$.[4] In figure 5, for a three player case, we can observe a direct consequence of this assumption: sequentially overlapped dominance regions. Therefore assumption A5 (PO) provides the necessary asymmetry in the game.

The last important remark about the assumptions is contained in the following lemma:

Lemma 2. $\exists\ \sigma_0 > 0$ *s.t.*$\forall \sigma \in (0, \sigma_0)$, $\forall j, i$ *if* $j > i$ *and* $x_j - x_i < \sigma$, *then* $\Delta\pi_i(n, x_i) - \Delta\pi_j(n, x_j) > 0\ \forall n.$

[4] Without loss of generality in the analysis we will assume the case where $\underline{k}_N < \overline{k}_1$, excluding the trivial situations where $\underline{k}_N > \overline{k}_1$, i.e. player N's lower dominance region does not overlap player 1's upper dominance region.

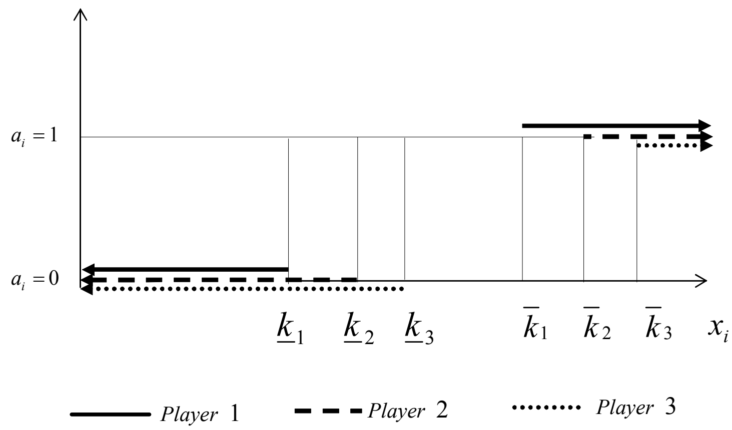

Figure 5: Overlapped Dominace Regions: Three Players Case

From assumption A5 (PO), we know that if two players face the same strategy profile and the same value of x, the "greater" player will get a lower net payoff. This lemma states that this is still true even when they face different values of x, such that their difference is less than σ_0.

3.1 Incomplete Information

Suppose now that the game is one of incomplete information in the payoff structure. Instead of observing the actual value of x, each player just observes a private signal x_i, which contains diffuse information about x. We assume that this is a game of private values, where each player gets utility directly from the signal rather than the actual value of the variable.[5]

The signal has the following structure: $x_i = x + \sigma\varepsilon_i$, where $\sigma > 0$ is a scale factor, x is drawn from the interval $[\underline{X}, \overline{X}]$ with uniform density, and ε_i is a random

[5]Even though we have not proven that our main result is robust to this assumption, it is simple to model the private value case as a limit of the common values case (when players derive utility from the actual value of the variable) as the noise goes to zero ($\sigma \to 0$). This approach has been used in the global game literature. (Carlsson and van Damme (1993), Morris and Shin (2000) and Frankel, Morris and Pauzner (2002).)

variable distributed according to a continuous density ϕ with support in the interval $[-\frac{1}{2}, \frac{1}{2}]$.We assume ε_i is $i.i.d.$ across the individuals.

This general noise structure has been used in the global game literature, allowing the conditional distribution of the opponents signal to be modelled in a simple way, i.e. given a player's own signal, the conditional distribution of an opponent's signal x_j admits a continuous density f_σ and a cdf F_σ with support in the interval $[x_i - \sigma, x_i + \sigma]$. Moreover this literature establishes a significant result: when the prior is uniform, players' posterior beliefs about the difference between their own observation and other players' observations are the same,[6] i.e. $F_\sigma(x_i \mid x_j) = 1 - F_\sigma(x_j \mid x_i)$.

In this context of incomplete information, a Bayesian pure strategy for a player i is a function $s_i : [\underline{X} - \frac{1}{2}\sigma, \overline{X} + \frac{1}{2}\sigma] \rightarrow A_i$, i.e. conditional on receiving a signal x_i player i takes an action $s_i(x_i) = a_i \in \{0, 1\}$. A pure strategy profile is denoted as $s = (s_1, s_{2,} ... s_N)$ where $s_i \in S_i$ and equivalently we define $s_{-i} = (s_1, s_2, ..s_{i-1}, s_{i+1}, ...s_N) \in S_{-i}$.

A switching strategy is a Bayesian pure strategy s_i satisfying : $\exists\ k_i\ s.t.$

$$s_i(x_i) = \begin{cases} 1 & if \quad x_i > k_i \\ 0 & if \quad x_i < k_i \end{cases}$$

Abusing notation, we write $s_i(\cdot; k_i)$ to denote the switching strategy with switching threshold k_i.

In this context of incomplete information, player i's payoff is characterized by his beliefs about his opponents strategies. In general, if player i is observing a signal x_i and is facing a strategy s_{-i} his expected net gain of choosing action 1 instead of action 0 can be written as

[6] This property holds approximately when x is not distributed with uniform density but σ is small, i.e. $F(x_i \mid x_j) \approx 1 - F(x_j \mid x_i)$ as σgoes to zero. See details in Lemma 4.1 Carlsson and van Damme (1993).

$$\Delta\Pi_i(s_{-i}, x_i) = \int_{x_{-i}} \Delta\pi_i(s_{-i}(x_{-i}), x_i) dF_{\sigma(-i)}(x_{-i} \mid x_i)$$

Calling this game of incomplete information $G_N(\sigma)$, let us define $BNE(G_N(\sigma))$ as the set of Bayesian Nash equilibria of $G_N(\sigma)$. The main result of the paper will prove that $G_N(\sigma)$ has a unique profile played in equilibrium as σ goes to 0. In this profile, every player will play a switching strategy $s_i(\cdot; x_i^*)$ where the threshold x_i^* solve the following equation:

$$\Delta\pi_i(i-1, x_i^*) = 0 \tag{1}$$

This states that, player i will switch from 0 to 1 at x_i^*, where x_i^* is the indifference point, when he faces a strategy profile such that all the players "lower" than him play action 1 and all the "higher" players play action 0. From lemma 2 we know that for all i, x_i^* not only exists, but it is also unique.

Let s^* be the profile such that each player is using a switching strategy $s_i(\cdot; x_i^*)$. The main result of the paper is the following theorem:

Theorem. *Consider a game $G_N(\sigma)$ satisfying assumptions (A1) to (A5), then $\exists\, \overline{\sigma} > 0$ s.t. $\forall \sigma \in (0,\overline{\sigma})$, $BNE(G_N(\sigma)) = \{s^*\}$.*

This proposition allows us to analyze a wide class of games of strategic substitutes where multiplicity is a problem, extending the global game literature. In particular this proposition generalizes the analysis and conclusion developed in the public good example of section 2; now, lower cost players are represented by a "higher" position in the players order (according to A5 (PO)), and they will switch between the actions at a higher threshold.

As an example, in figure 6 we show a three players case. The strategy profile in equilibrium shows the higher player switching at the beginning of his upper dominance

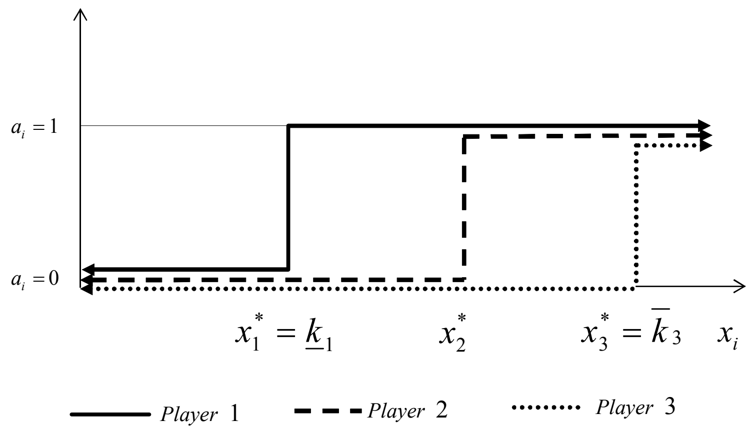

Figure 6: Equilibrium Selection: Three Players Case

region, $x_3^* = \overline{k}_3$. The lower player switches at the end of his lower dominance region $x_1^* = \underline{k}_1$, and player 2 switches at x_2^* where $\underline{k}_1 < x_2^* < \overline{k}_3$.

4 Proof of the Theorem

In this section we develop the main steps of the proof of the theorem. We will argue that the profile s^* is the unique profile surviving a particular process of iterated deletion of strictly dominates strategies. We start defining the sequence of undominated sets. Note however, that these are not the standard undominated sets used to define iteratively undominated strategies. Instead these are sets defined by an alternative process that eliminates profiles that are not part of any equilibrium. These strategies are strictly dominated when we restrict ourselves to considering some subset of others players' actions that are "potentially" part of some Nash equilibrium profile. We call these sets the *conditionally iteratively undominated sets*.[7]

We will prove that this process does not rule out any Bayesian Nash equilibria.

[7] Since the elimination proceeds upon players receiving the signal, then formally these sets contain strategies that are *interim* strictly undominated.

We then proceed to show that, under our mentioned assumptions the strategy profile surviving the iterated deletion is unique. Hence only one equilibrium survives. We now describe the structure of the conditionally undominated sets in $G_N(\sigma)$ satisfying assumption A1 to A5, and then proceed to give a formal proof of the theorem.

4.1 The Conditionally Undominated Profiles

For a given game $G_N(\sigma)$, let us define the process of deletion such that any strategy profile that survives t rounds of iterated elimination of conditionally strictly dominated strategies is contained in S^t, where $S^t = \underset{i=1}{\overset{N}{\times}} S_i^t \ \forall t$ (and $S_{-i}^t = \underset{j \neq i}{\times} S_j^t \ \forall t \ \forall i$). If player i' best response correspondence is defined as $BR_i(s_{-i}) = \{s_i \in S_i : \Pi_i(s_i, s_{-i}, x_i) \geq \Pi_i(s_i^{'}, s_{-i}, x_i) \ \forall x_i \ \forall s_i^{'} \in S_i\}$, then the conditionally undominated sequence $\{S^t\}_{t=0}^{\infty}$ is defined as follows:

Set $S_i^0 \equiv S_i$ and $\widetilde{S}_i^0 \equiv S_{-i}$, then $\forall t > 0$

$$\widetilde{S}_{-i}^t \equiv \left\{\widetilde{s}_{-i} \in S_{-i}^t : \exists \widetilde{s}_i \in S_i^{t-1} \ s.t. \ \widetilde{s}_j = BR_j(\widetilde{s}_{-j}) \ \forall j \neq i\right\} \tag{2}$$

and

$$S_i^t \equiv \left\{ \begin{array}{c} s_i \in S_i^{t-1} : \ \nexists s_i^{'} \in S_i^{t-1} \ such \ that \\ \Pi_i(s_i^{'}, s_{-i}, x_i) \geq \Pi_i(s_i, s_{-i}, x_i) \ \forall x_i \ \forall s_{-i} \in \widetilde{S}_{-i}^{t-1} \\ and \ with \ strict \ inequality \ for \ some \, x_i \end{array} \right\} \tag{3}$$

This states that, $\widetilde{S}_{-i}^{t-1}$ is the set of all others players' strategy profiles that, for some strategy $s_i \in S_i^{t-1}$, contains strategies that are mutually best responses (excluding player i). Recall from section 3.1, that $\Pi_i(a_i, s_{-i}, x_i)$ represents player i's expected payoff when, upon observing a signal x_i and facing a strategy s_{-i}, he chooses action

a_i. Therefore in each round, all the strategies that are strictly dominated when opponents actions are restricted to those that are "potentially" part of a Bayesian Nash equilibrium profile, are eliminated.

The following lemma establishes an important characteristic: For a general game $G_N(\sigma)$ the conditional iterative process of elimination of strategies described above, does not rule out any Bayesian Nash equilibrium.

Lemma 3. $S^t \supseteq BNE(G_N(\sigma))\ \forall t.$

Additionally, the following lemma proves that if game a $G_N(\sigma)$ satisfies assumptions (A1) to (A5) and, if for some t, player N´s set of undominated strategies contains a unique strategy such that he switches from action 0 to action 1 at $\overline{k}_N$, then there exist a unique undominated profile s^* in S^t. The profile s^* is a Bayesian Nash equilibrium such that each component is a switching strategy where the cutoff solves equation 1. More formally:

Lemma 4. *Consider a game $G_N(\sigma)$ satisfying assumptions (A1) to (A5). Suppose $\exists\, t$ such that $S_N^t = \{s_N^*\}$, then $S^t = \{s^*\}$ and $s^* \in BNE(G_N(\sigma))$.*

4.2 Iterated Elimination of Conditionally Dominated Strategies and Proof of the theorem.

Now we describe the structure of the undominated set S^t for a game $G_N(\sigma)$ satisfying assumption A1 to A5. Let us first define the sequences $\{\underline{x}_i^t\}_{t=0}^{\infty}\ \forall i$.

Set $\underline{x}_i^0 \equiv -\infty$, and for $t > 0$ each element of the sequence is calculated as follows:

$$\underline{x}_i^t \equiv \min_{s_{-i} \in \widetilde{S}_{-i}^{t-1}} \{x_i : \Delta\Pi_i(s_{-i}, x_i) = 0\}$$

Every element of the sequence represent the minimum signal among which the player is indifferent between the two actions, but now just considering all available strategies of his opponent that belong to the set $\widetilde{S}_{-i}^{t-1}$, i.e. considering just the strategy profiles that, for some strategy $s_i \in S_i^{t-1}$, contains strategies that are mutually best responses (excluding player i). Since $\widetilde{S}_{-i}^{t-1} \subseteq \widetilde{S}_{-i}^{t}$ it is easy to see that $\{\underline{x}_i^t\}_{t=0}^{\infty}$ is an increasing sequence.

Now, keeping in mind that x_i^* is the signal that solves equation 1 and describes player i' switching point in the profile of switching strategies s^*, the following lemma characterize the structure of every set S_i^t.

Lemma 5. *Consider a game $G_N(\sigma)$ satisfying assumptions (A1) to (A5). Then $\exists\, \overline{\sigma} > 0$ s.t. $\forall \sigma \in (0, \overline{\sigma})$ the conditionally undominated sequence $\{S^t\}_{t=0}^{\infty}$ satisfies:*

i) $\forall t\ \forall i$

$S_i^t = \{s_i : s_i(x_i) = 0 \ if \ x_i < \min\{\underline{x}_i^t, x_i^*\} \ \ and \ s_i(x_i) = \ 1 \ if \ x_i \in (x_i^*, \underline{x}_i^t) \cup (\overline{k}_N, \overline{X} + \sigma)\}$

and if $i > j$ then $\underline{x}_i^t > \underline{x}_j^t$.

ii) $\exists\, t'$ s.t. $\forall\, t \geq t'$, $\underline{x}_N^t = \overline{k}_N$

The first part of this lemma describe the structure of every strategy surviving iterated deletion of conditionally dominated strategies. It shows that, giving $\underline{x}_i^t$, every strategy in S_i^t plays action 0 for signals less than the minimum between $\underline{x}_i^t$, x_i^*, and plays action 1 for a signals in his upper dominance region and for signal in the interval $(x_i^*, \underline{x}_i^t)$. However notice that $(x_i^*, \underline{x}_i^t) = \phi$ if $\underline{x}_i^t \leq x_i^*$. This lemma also establishes that the greater the player (according to assumption A5) the greater the value of $\underline{x}_i^t$, i.e. the sequences $\{\underline{x}_i^t\}_{t=0}^{\infty}$ preserve players' order.

The second part of the lemma state that player N' sequence $\{\underline{x}_i^t\}_{t=0}^{\infty}$, reaches his upper dominance region in a finite number of steps.

In figures 7 and 8 we illustrate the structure of the surviving strategies for the

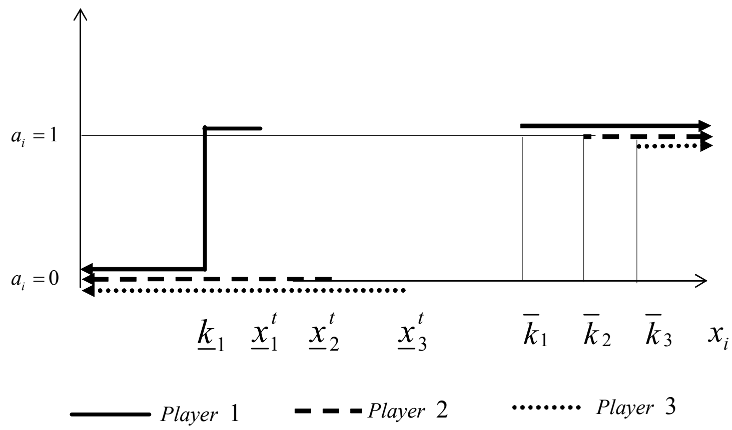

Figure 7: Case $\underline{x}_2^t < x_2^*$

three player case. Figure 7 shows the case when $\underline{x}_2^t < x_2^*$ and figure 8 shows the case when $\underline{x}_2^t \geq x_2^*$.

Having described and characterized the conditionally undominated sequence $\{S^t\}_{t=0}^{\infty}$ for any game $G_N(\sigma)$, we next state the theorem again and develop the proof.

Theorem. *Consider a game $G_N(\sigma)$ satisfying assumptions (A1) to (A5), then* $\exists\, \overline{\sigma} > 0$ *s.t.* $\forall \sigma \in (0,\overline{\sigma})$, $BNE(G_N(\sigma)) = \{s^*\}$.

Proof. From Lemma 6 it follows directly that $\exists\, \overline{\sigma} > 0$ *s.t. for all* $\sigma \in (0,\overline{\sigma})$ and $\forall\, t \geq t'$ $S_N^t = \{s_N^*\}$. Therefore using Lemma 5 we can conclude that $\forall\, t \geq t'$ $S^t = \{s^*\}$. Finally from Lemma 4 we know that $S^t \supseteq BNE(G_N(\sigma))$ then $BNE(G_N(\sigma)) = \{s^*\}$ ■

5 Conclusions

The global game approach is a proven method to incorporate more realistic assumptions in game-theoretic models. Assuming a very general payoff structure, the approach examines Nash equilibria as a limit of equilibria of payoff-perturbed games.

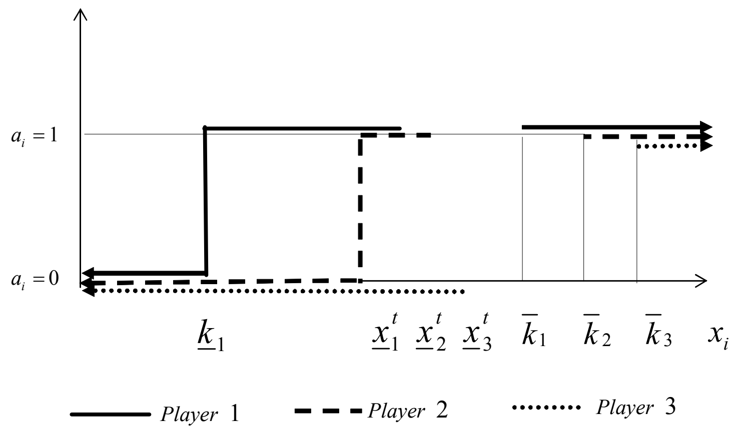

Figure 8: Case $\underline{x}_2^t \geq x_2^*$

Carlsson and van Damme (1993) show that in binary action two-player games, there exists a unique equilibrium profile surviving iterated deletion of strictly dominated strategies. Recently this result has been generalized by Frankel, Morris and Pauzner (2002) to many players and actions, but limiting the analysis to games with strategic complementarities.

Continuing with this line of research, we extend the literature proving an equilibrium selection result for a class of global games with strategic substitutes. Assuming a particular asymmetry in the players' dominance regions, we prove that for a general class of binary action, N-player games, each such game has a unique equilibrium strategy profile. This result might allows us to analyze a wide class of games of strategic substitutes such as collective action problems, entry-exit models in industrial organization etc. In particular we apply the result to a model of public good provision. The interesting conclusion to this application is that the equilibrium profile induces an efficient provision of the public good, and the contributions come from the lowest cost contributors. In general the result provides a useful tool for applications.

Further research must be devoted to extend the result to games with more than two actions and with common values, i.e. where players derive utility from the actual value of the exogenous parameter rather than the signal.

6 Appendix

To ease the exposition, we now repeat the formal statement of the proposition and lemmas 1 to 5 before each of their proofs.

Proposition. *Consider a game* $M_2(\sigma)$ *satisfying assumptions (a1) to (a4). There exists a unique strategy profile* s^* *that survives iterated deletion of the strictly dominated strategies for a sufficient small amount of noise, so that* $\exists\ \overline{\sigma} > 0$, *s.t.* $\forall\sigma \in (0,\overline{\sigma})$, $BNE(M_2(\sigma)) = \{s^*\}$.

Proof. Application of the theorem page 996, Carlsson and van Damme (1993).■

Lemma 1. *There exists a unique* $\widetilde{x} \in [\underline{X}, \overline{X}]$ *solving* $\Delta\pi_i(n, \widetilde{x}) = 0$.
Therefore $\Delta\pi_i(n, x) < 0\ \forall\, x < \widetilde{x}$ *and* $\Delta\pi_i(n, x) > 0\ \forall\, x > \widetilde{x}\ \forall i\ \forall n$.

Proof . Since $\Delta\pi_i(\cdot, x)$ is continuous and monotonic (assumption A2 (C) and A3(M)), $\exists!\ \widetilde{x}\ s.t.\ \Delta\pi_i(n, \widetilde{x}) = 0$ and $\Delta\pi_i(n, x) < 0\quad for\ all\ \ x < \widetilde{x}$ and $\Delta\pi_i(n, x) > 0\quad for\ all\ \ x > \widetilde{x}$. By assumption A4 we know that $\widetilde{x} \in [\underline{X}, \overline{X}]$ for $n = 0$ and for $n = N - 1$.Therefore by strategic substitutes (A1), $\widetilde{x} \in [\underline{X}, \overline{X}]\ \forall n$.■

Lemma 2. $\exists\ \sigma_0 > 0\ \ s.t. \forall\sigma \in\ (0, \sigma_0),\ \forall j, i\quad if\ \ j > i\ \ and\ \ x_j - x_i < \sigma$, *then* $\Delta\pi_i(n, x_i) - \Delta\pi_j(n, x_j) > 0\ \forall n$.

Proof . From assumption A5 (PO) we know that there exists a players order $\{1, ..., N\}$ such that $\exists\ \alpha > 0\quad s.t$ if $j > i$ then $\Delta\pi_i(n, x) - \Delta\pi_j(n, x) > \alpha\ \forall i, j\ \forall n$. Hence using assumption A3 (M) we know that $\forall j \neq i$ if $x_i < x_j\ \exists\ \sigma'_{ji} > 0$ $s.t.\ \Delta\pi_i(n, x_i) - \Delta\pi_j(n, x_i + \sigma'_{ji}) = 0\ \forall n$. Let $\sigma_0 \equiv \min\{\sigma'_{ji}\}_{j\neq i}$ therefore $\forall\sigma \in\ (0, \sigma_0)$ if $j > i$ and $x_j - x_i < \sigma$ then $\Delta\pi_i(n, x_i) - \Delta\pi_j(n, x_j) > 0\ \forall n$.■

Lemma 3 $S^t \supseteq BNE(G_N(\sigma))\ \forall t$.

Proof . By contradiction let us suppose that $S^t \subset BNE(G_N(\sigma))$ for some t.

Then there exists a profile $s \in BNE(G_N(\sigma))$ and $s \notin S^t$ for some t.

Since $s \in BNE(G_N(\sigma))$ implies

$\Pi_i(s_i(x_i), s_{-i}, x_i) \geq \Pi_i(s_i^{'}(x_i), s_{-i}, x_i)\ \ \forall x_i\ \forall s_i^{'} \in S_i$ and $\forall i$

but $s \notin S^t(\underline{x}^t)$ then $\exists\ s_i^{'} \in S_i^{t-1}$ *s.t.* $\Pi(s_i^{'}(x_i), s_{-i}, x_i) > \Pi(s_i(x_i), s_{-i}, x_i)$ for some x_i and for some $s_{-i} \in \widetilde{S}_{-i}^{t-1}$. Therefore s is not a Bayesian Nash equilibrium profile. Hence it must be the case that $S^t \supseteq BNE(G_N(\sigma))\ \forall t$.■

In order to develop proofs for lemmas 4 and 5 we first need to introduce the notion of *reduced game* and *extremal profiles.* We also state some of their properties.

Definition 1. **Reduced Game:** Consider a game $G_N(\sigma)$ as defined in section 3, and an arbitrary subset of players I. Let $s_I = (s_i)_{i \in I}$ and $s_{-I} = (s_i)_{i \notin I}$. Conditionally on s_{-I}, we define $G_I(\sigma, s_{-I})$ as a *reduced game* (with I players) of the original game $G_N(\sigma)$. It is easy to check that if $G_N(\sigma)$ satisfies assumptions A1 A2, A3, A5, the same assumptions hold for the reduced game $G_I(\sigma, s_{-I})$. Additionally, if conditionally on s_{-I} there exists an interval of signal $[\underline{\lambda}, \overline{\lambda}] \subseteq [\underline{X}, \overline{X}]$ such that for every player i $\in I$, there exist upper and lower dominance regions (according to assumption A4), then $G_I(\sigma, s_{-I})$ is a reduced game that holds the same properties of the original game $G_N(\sigma)$. These fact may allow us to use results from games with less players.

Definition 2. **Extremal Profiles:** Let $\overline{s}_{-i}^{t-1}$, $\underline{s}_{-i}^{t-1}$ be the extremal profiles for some player i. These two particular profiles are defined as follows:

$\overline{s}_{-i}^{t-1} = \arg\max_{s_{-i} \in S_{-i}^{t-1}} \Delta\Pi_i(s_{-i}, x_i)$

$\underline{s}_{-i}^{t-1} = \arg\min_{s_{-i} \in S_{-i}^{t-1}} \Delta\Pi_i(s_{-i}, x_i)$

In words,[8] by strategic substitutes (A1 (SS)) if player i, upon receiving a signal

[8]Notice that by strategic substitutes (assumption A1), that both $\overline{s}_{-i}^{t-1}$ and $\underline{s}_{-i}^{t-1}$ are determined independently of the value of x_i.

x_i and assuming that his opponents are using the strategy profile $\overline{s}_{-i}^{t-1}$ ($\underline{s}_{-i}^{t-1}$), chooses action 0 (1) he will choose action 0 (1) for all $s_{-i} \in S_{-i}^{t-1}$.

Analogously define $\overline{s}_{-i}'^{t-1}$, $\underline{s}_{-i}'^{t-1}$ as the extremal profiles restricted to $\widetilde{S}_{-i}^{t-1}$, where recall $\widetilde{S}_{-i}^{t-1}$ is defined by equation 2. More formally:

$$\overline{s}_{-i}'^{t-1} \in \underset{s_{-i} \in \widetilde{S}_{-i}^{t-1}}{\arg\max} \Delta\Pi_i(s_{-i}, x_i)$$

$$\underline{s}_{-i}'^{t-1} \in \underset{s_{-i} \in \widetilde{S}_{-i}^{t-1}}{\arg\min} \Delta\Pi_i(s_{-i}, x_i)$$

Therefore, since by definition $S_{-i}^{t-1} \supseteq \widetilde{S}_{-i}^{t-1}$, by strategic substitutes the following inequalities holds for all $s_{-i} \in \widetilde{S}_{-i}^{t}$:

$$\begin{aligned} \Delta\Pi_i(\overline{s}_{-i}^{t-1}, x_i) &\geq \Delta\Pi_i(\overline{s}_{-i}'^{t-1}, x_i) \geq \Delta\Pi_i(s_{-i}, x_i) \text{ and} \\ \Delta\Pi_i(s_{-i}, x_i) &\geq \Delta\Pi_i(\underline{s}_{-i}'^{t-1}, x_i) \geq \Delta\Pi_i(\underline{s}_{-i}^{t-1}, x_i). \end{aligned}$$

Lemma 4. *Consider a game $G_N(\sigma)$ satisfying assumptions (A1) to (A5). Suppose $\exists\ t$ such that $S_N^t = \{s_N^*\}$, then $S^t = \{s^*\}$ and $s^* \in BNE(G_N(\sigma))$.*

Proof. If player N plays the strategy $s_N(\cdot; \overline{k}_N)$, i.e.

$$s_N(x_N; \overline{k}_N) = \begin{cases} 1 & if \quad x_N > \overline{k}_N \\ 0 & if \quad x_N < \overline{k}_N \end{cases}$$

Then the subset of players $\{1...N-1\}$ face a reduced game $G_{N-1}(\sigma, s_N^*)$. For the subset of signal $[\underline{X} - \sigma, \overline{k}_N - \sigma]$ it is also easy to check that $G_{N-1}(\sigma, s_N^*)$ satisfies assumption A1 to A5. From Lemma 5.A (stated at the end of this appendix), or applying the Carlsson and van Damme result,[9] we know that any game $G_2(\sigma)$ satisfying assumptions A1 to A5 has the equilibrium structure according to equation 1. Then by induction we can assume that $G_{N-1}(\sigma, s_N^*)$ has a unique equilibrium also according equation 1. Therefore $S^t = \{s^*\}$.

Using the same argument, but starting from the right hand side, it easy to check that the unique best response for player N to the profile s_{-N}^* is to play s_N^*.■

We prove next lemma 5. We will develop the proof using an induction argument in the number of players, then in order to ease the exposition we first present a version of lemma 5 but for the two player case. We call this previous lemma, lemma 5.A.

Lemma 5.A. Consider a game $G_2(\sigma)$ satisfying assumptions A1 to A5. Then $\exists\, \overline{\sigma} > 0\ s.t.\ \forall \sigma \in (0,\overline{\sigma})$ the conditionally undominated sequence $\{S^t\}_{t=0}^{\infty}$ satisfies:

i) $\forall t\ \forall i$

$S_1^t = \{s_1 : s_1(x_1) = 0\ if\ x_1 < x_1 < \underline{k}_1\ \ and\ s_i(x_i) = 1\ if\ x_i \in (\underline{k}_1, \underline{x}_1^t) \cup (\overline{k}_1, \overline{X})\}$

$S_2^t = \{s_2 : s_2(x_2) = 0\ if\ x_2 < \underline{x}_2^t\ \ and\ s_i(x_i) =\ 1\ if\ x_2 > \overline{k}_2\}$

and if then $\underline{x}_2^t > \underline{x}_1^t$.

ii) $\exists\, t'\ s.t.\ \forall\, t \geq t',\ \underline{x}_2^t = \overline{k}_2$

Proof. Part i). First let us define $\overline{\sigma} \equiv \min\left\{(\underline{k}_2 - \underline{k}_1), (\overline{k}_2 - \overline{k}_1), \sigma_0\right\}$, where σ_0 is defined according to Lemma 3. Now, from A1 (SS) if s_1 is a best response (BR) to a switching strategy $s_2(\cdot; \underline{k}_2)$, it will be a BR to any $s_2 \in S_2^0$, then it is easy to check that

$$\Delta\Pi_1(s_2(x_2; \underline{k}_2), x_1 = \underline{k}_2 - \sigma) = \Delta\pi_1(0, \underline{k}_2 - \sigma) > 0$$

$$\Delta\Pi_1(s_2(x_2; \underline{k}_2), x_1 = \underline{k}_2 + \sigma) = \Delta\pi_1(1, \underline{k}_2 + \sigma) < 0$$

[9] See theorem page 996, Carlsson and van Damme (1993)

So, given continuity of the payoff function we can use the intermediate value theorem: $\exists\ \underline{x}_1^1 > \underline{k}_1$, where $\underline{x}_1^1 = \min\{x_1 : \Delta\Pi_1(s_2(\cdot;\underline{k}_2), x_1) = 0\}$ and[10]

$$S_1^1 = \left\{s_1 \in S_1^0 : s_1(x_1) = 0\ if\ x_1 < \underline{k}_1\ and\ s_1(x_1) = 1\ if\ x_1 \in (\underline{k}_1, \underline{x}_1^1) \cup (\overline{k}_1, \infty)\right\}$$

Now, player 2's first round of elimination proceeds as follows: by A1 (SS), if $s_2 \in S_2^1$ is a BR to $s_1^-(\cdot;\underline{x}_1^1)$, where $s_i^-(\cdot;x_i)$ to denote the "inverse" switching strategy, which switches from 1 to 0 at x_i, it will be a BR to any $s_1 \in S_1^1$. Then the net payoff of player 2 observing a signal $x_2 = \underline{k}_2$ and facing a strategy $s_1^-(\cdot;\underline{x}_1^1)$ is

$$\Delta\Pi_2(s_1^-(\cdot;\underline{x}_1^1), x_2 = \underline{k}_2) = \Delta\pi_2(1,\underline{k}_2)F_\sigma(\underline{x}_1^1 \mid \underline{k}_2) + \Delta\pi_2(0,\underline{k}_2)(1 - F_\sigma(\underline{x}_1^1 \mid \underline{k}_2))$$

From A4 (IS) $\Delta\pi_2(0,\underline{k}_2) = 0$ and from A1 (SS) $\Delta\pi_2(1,\underline{k}_2) < 0$. Since $0 < F_\sigma(\underline{x}_1^1 \mid \underline{k}_2) < 1$, therefore $\Delta\Pi_2(s_1^-(\cdot;\underline{x}_1^1), \underline{k}_2) < 0$.

Now, by A1 (SS) $\Delta\Pi_2(s_1^-(\cdot;\underline{x}_1^1), x_2 = \underline{x}_1^1 + \sigma) = \Delta\pi_2(0, x_2) > 0$. Again, given continuity of the payoff function we can use the intermediate value theorem: $\exists\ \underline{x}_2^1 > \underline{k}_2$, where $\underline{x}_2^1 = \min\left\{x_2 : \Delta\Pi_2(s_1^-(\cdot;\underline{x}_1^1), x_2) = 0\right\}$, and

$$S_2^1 \equiv \left\{s_2 \in S_2^0 : s_2(x_2) = 0\ if\ x_2 < \underline{x}_2^1\ and\ s_2(x_2) = 1\ if\ x_2 > \overline{k}_2\right\}$$

Repeating this process of iteration it is easy to prove by induction that a strategy profile s surviving t rounds of elimination is contained in the set S^t such that:

$S_1^t : \{s_1 : s_1(x_1) = 0\ if\ x_1 < \underline{k}_1\ and\ s_1(x_1) = 1\ if\ x_1 \in (\underline{k}_1, \underline{x}_1^t) \cup (\overline{k}_1, \overline{X})\}$

$S_2^t : \{s_2 : s_2(x_2) = 0\ if\ x_2 < \underline{x}_2^t\ and\ s_2(x_2) = 1\ if\ x_2 > \overline{k}_2\}$

[10] Notice that by construction $0 < \underline{k}_2 - \underline{x}_1^1 < \sigma$

and $\underline{x}_1^t,\ \underline{x}_2^t$ are obtained recursively from the following equations

$$\underline{x}_1^t = \min\left\{x_1 : \Delta\Pi_1(s_2(\cdot;\underline{x}_2^{t-1}), x_1) = 0\right\} \tag{4}$$

$$\underline{x}_2^t = \min\left\{x_2 : \Delta\Pi_2(s_1^-(\cdot;\underline{x}_1^t), x_2) = 0\right\} \tag{5}$$

Finally, notice that the process governed by equations 4 and 5 defines two strictly increasing sequences $\{\underline{x}_1^t\}_,\ \{\underline{x}_2^t\}$ where $\underline{x}_2^t > \underline{x}_1^t\ \forall t$.

Part *ii*). Now, let us suppose now that there exist the limit point $\underline{x}_1^\infty$ and by construction there also exists $\underline{x}_2^\infty$ where $0 \le \underline{x}_2^\infty - \underline{x}_1^\infty < \sigma$. Rewriting the conditions of equations 4 and 5 and using the equivalence $F_\sigma(\underline{x}_1^\infty \mid \underline{x}_2^\infty) = 1 - F_\sigma(\underline{x}_1^\infty \mid \underline{x}_2^\infty)$ we get

$$\Delta\pi_1(1,\underline{x}_1^\infty)F_\sigma(\underline{x}_1^\infty \mid \underline{x}_2^\infty) + \Delta\pi_1(0,\underline{x}_1^\infty)(1 - F_\sigma(\underline{x}_1^\infty \mid \underline{x}_2^\infty)) = 0$$

$$\Delta\pi_2(1,\underline{x}_2^\infty)F_\sigma(\underline{x}_1^\infty \mid \underline{x}_2^\infty) + \Delta\pi_2(0,\underline{x}_2^\infty)(1 - F_\sigma(\underline{x}_1^\infty \mid \underline{x}_2^\infty)) = 0 \tag{6}$$

It must be also true that the difference between these two equations is zero as well, so

$$\begin{gathered}(\Delta\pi_1(1,\underline{x}_1^\infty) - \Delta\pi_2(1,\underline{x}_2^\infty))F_\sigma(\underline{x}_1^\infty \mid \underline{x}_2^\infty) + \\ (\Delta\pi_1(0,\underline{x}_1^\infty) - \Delta\pi_2(0,\underline{x}_2^\infty))(1 - F_\sigma(\underline{x}_1^\infty \mid \underline{x}_2^\infty)) = 0\end{gathered} \tag{7}$$

but from lemma 3 and since $0 < F_\sigma(\underline{x}_1^\infty \mid \underline{x}_2^\infty) < 1$ we know that each term in equation 7 is strictly positive, then $\Delta\Pi_1(s_2(\cdot;\underline{x}_2^\infty),\underline{x}_1^\infty) - \Delta\Pi_2(s_1^-(\cdot;\underline{x}_1),\underline{x}_2^\infty) > 0$. Contradiction. Hence, since $\{\underline{x}_1^t\}$ is an increasing unbounded sequence it must be the case that $\exists\ t^*\ s.t.\ \forall t > t^*\ \underline{x}_1^t \ge \overline{k}_1$. i.e. in a finite number of steps the sequence reaches the upper dominance region eliminating all strategies but one: $s_1(\cdot;\underline{k}_1)$. On the other hand $\underline{x}_2^{t^*} < \overline{k}_2$ which implies that $\Delta\Pi_2(s_1(\cdot;\underline{k}_1), \underline{x}_2^{t^*}) = \Delta\pi_2(1, \underline{x}_2^{t^*}) < 0$, therefore $\underline{x}_2^{t^*+1} = \overline{k}_2$ is the last expected payoff where he is indifferent between the two actions: $\Delta\Pi(s_1(\cdot;\underline{k}_1), \overline{k}_2) = \Delta\pi_2(1, \overline{k}_2) = 0$. Finally set $t' = t^* + 1$ then $\forall\, t \geq t'$, $\underline{x}_2^t = \overline{k}_2$.$\blacksquare$

Lemma 5. *Consider a game $G_N(\sigma)$ satisfying assumptions (A1) to (A5). Then $\exists\, \overline{\sigma} > 0$ s.t. $\forall \sigma \in (0, \overline{\sigma})$ the conditionally undominated sequence $\{S^t\}_{t=0}^{\infty}$ satisfies:*

i) $\forall t\ \forall i$

$S_i^t = \{s_i : s_i(x_i) = 0\ if\ x_i < \min\{\underline{x}_i^t, x_i^*\}\ \ and\ s_i(x_i) =\ 1\ if\ x_i \in (x_i^*, \underline{x}_i^t) \cup (\overline{k}_N, \overline{X} + \sigma)\}$

and if $i > j$ then $\underline{x}_i^t > \underline{x}_j^t$.

ii) $\exists\, t'$ s.t. $\forall\, t \geq t'$, $\underline{x}_N^t = \overline{k}_N$

Proof. First define $\overline{\sigma} = \min\left\{(\underline{k}_i - \underline{k}_{i-1})_{i=2}^N, (\overline{k}_i - \overline{k}_{i-1})_{i=2}^N, \sigma_0\right\}$, where σ_0 is calculated according to Lemma 2. Second, since we need to prove that both parts of the lemma are true for all i and for all t, let us introduce an induction argument in the number of players. Lemma 5.A shows that Lemma 5 is true for games with two players $G_2(\sigma)$ satisfying assumptions A1 to A5. Therefore let us assume that lemma 5 remain valid for games with $N-1$ players satisfying assumptions A1 to A5.

Now, having assumed the inductive process in the number of players, we will prove the first part of Lemma 5 through induction in t and we will prove the second part showing that $G_N(\sigma)$ is "composed" of two reduced games.

Proof of part i). First consider the first round of conditional elimination, $t = 1$. By definition of σ, we know that $\underline{k}_{N-1} < \underline{k}_N, -\sigma$ then,[11] $\forall i = 1,..N-1$ and $\forall x_i < \underline{k}_N - \sigma$ player i's payoff is $\Delta\Pi_i(s_{-\{i,N\}}, s_N = 0)$.Then, the subset of players $\{1...N-1\}$ face a reduced game $G_{N-1}(\sigma, s_N = 0)$. It is easy to check that for signals

[11] Recall that $\underline{k}_{N-1} > ... > \underline{k}_1$

in the subset $[\underline{X}-\sigma, \underline{k}_N-\sigma]$, the reduce game $G_{N-1}(\sigma, s_N=0)$ satisfies assumptions A1 to A5. By the induction assumption we know that Lemma 5 holds for games with $N-1$ players, then for players 1 to $N-1$ the first round of elimination coincide with the one in the reduced game. More formally: for every player in $G_{N-1}(\sigma, s_N=0)$ there exist $\underline{\widehat{x}}_i^1$ and $\widehat{S}_i$, hence $\underline{x}_i^1 = \underline{\widehat{x}}_i^1$ and $S_i^1 = \widehat{S}_i^1$. Now, since $\underline{x}_{N-1}^1 < \underline{k}_N - \sigma$, then the minimum signal for which player N is indifferent between the two actions is $\underline{k}_N$, i.e. $\underline{x}_N^1 = \underline{k}_N$. Therefore the undominated set S_N^1 contain all strategies that plays the dominant action in the corresponding dominant region, i.e. $S_N^1 = \{s_N : s_N(x_N) = 0\ if\ x_N < \underline{x}_N^1\ and\ 1\ if\ x_N > \overline{k}_N\}$.

Now, following the induction argument in t, we assume that lemma 5 is true for a round $t-1$. Let us divide the analysis in to possible cases:

a) if $\underline{x}_{N-1}^{t-1} < x_{N-1}^*$

b) if $\underline{x}_{N-1}^{t-1} \geq x_{N-1}^*$

and let us show that the lemma is true in both cases.

a) First, without loss of generality let assume that some player $l \in \{1, ..., N-1\}$ is the "last" player to reach his threshold x_l^*. Therefore the induction assumption can be states as follows:

$\forall i = 1, ..., l$

$S_i^{t-1} = \{s_i : s_i(x_i) = 0\ if\ x_i < x_i^*\ \ and\ \ 1\ if\ x_i \in (x_i^*, \underline{x}_i^{t-1}) \cup (\overline{k}_i, \overline{X})\}$

$\forall i = (l+1), ..., N$

$S_i^{t-1} = \{s_i : s_i(x_i) = 0\ if\ x_i < \underline{x}_i^{t-1}\ \ and\ \ 1\ if\ x_i > \overline{k}_i\}$,

and if $i > j$ then $\underline{x}_i^{t-1} > \underline{x}_j^{t-1}$

Now it is enough to prove that S^t has the same structure as S^{t-1}, i.e players $i = (l+1), ..., N-1$ will increment the set of signals where they pick action 0 and players $i = 1, ..., l$ will increment the set of signals where they pick action 1.

By strategic substitutes, for all $i = (l+1), ..., N-1$ the component of the extreme profile $\overline{s}_{-i}^{t-1}$, associated with the N^{th} player is $s_N(\cdot;\overline{k}_N)$. Then if player i, upon receiving a signal x_i and assuming that his opponents are using the strategy profile $\overline{s}_{-i}^{t-1}$, chooses action 0 he will choose action 0 for all $s_{-i} \in \widetilde{S}_{-i}^{t-1}$.

Recall that if players $i = 1, ..., l$ choose action 1 when other players' strategy is $\underline{s}_{-i}^{\prime t-1}$, they will pick action 1 for all $s_{-1} \in \widetilde{S}_{-i}^{t}$. We now prove that $\forall i \in \{1, ..., l\}$ and $\forall x_i < x_{N-1}^*$, the component, of the extreme profile $\underline{s}_{-i}^{\prime t-1}$, associated with the N^{th} player is $s_N(\cdot;\overline{k}_N)$ as well. Suppose by way of contradiction that the strategy is different from $S_N(\cdot;\overline{k}_N))$ for some $x_i < x_{N-1}^*$. Then $\exists$ a neighborhood $O \subseteq [x_i - \sigma, x_i + \sigma]$ such that on O $s_N(x_N) = 1$. Since for signals less than x_{N-1}^* we cannot have the $N-1$ dimensional profile $(1, ..., 1)$ on O. By anonymity permute player N with some player j $\underline{s}_j^{\prime t-1}(x_j) = 0$, then by strategic substitutes this permuted strategies lower payoff $\Delta\Pi_i$, and so $\underline{s}_{-i}^{\prime t-1}$ could not have been a minimizer of $\Delta\Pi_i$ on $\widetilde{S}_{-i}^{t-1}$.

Now, we conclude that, for all signal less than x_{N-1}^* the extremal profiles $\overline{s}_{-i}^{t-1}$ $\forall i \in \{(l+1), ..., N-1\}$and $\underline{s}_{-i}^{\prime t-1}$ $\forall i \in \{1, ..., l\}$ coincide at least in their last component; both profiles consider player N playing the switching strategy $s_N(\cdot;\overline{k}_N)$. Then the subset of players $\{1..., N-1\}$ face a reduced game $G_{N-1}(\sigma, s_N = 0)$. It is easy to check that for the subset of signal $[\underline{X} - \sigma, x_{N-1}^*]$, the reduced game $G_{N-1}(\sigma, s_N = 0)$ satisfies assumptions A1 to A5. By the induction assumption we know that Lemma 5 holds for games with $N-1$ players, then for players 1 to $N-1$ the round of elimination t coincides with the same round of the reduced game. More formally: for every player in $G_{N-1}(\sigma, s_N = 0)$ there exist $\underline{\widehat{x}}_i^t$ and $\widehat{S}_i^t$, hence $\underline{x}_i^t = \underline{\widehat{x}}_i^t$ and $S_i^t = \widehat{S}_i^t$, i.e. S_i^t have the same structure than S_i^{t-1} for all $i \in \{1, ..., N-1\}$.

By assumption A5 (Players Order) and using the intermediate value theorem it is easy to check that there exists $\underline{x}_N^t > \underline{x}_{N-1}^t$ such that $\Delta\Pi_N(\overline{s}_{-N}^{t-1}, \underline{x}_N^t) = 0$. Therefore $S_N^t = \{s_N : s_N(x_N) = 0 \; if \; x_N < \underline{x}_N^t \; and \; s_N(x_N) = 1 \; if \; x_N > \overline{k}_N\}$

b) In this second case we assume that $\underline{x}_{N-1}^{t-1} \geq x_{N-1}^*$. We will follow the same argument described and used in the previous part. Now, since we are considering signals such that $\underline{x}_{N-1}^{t-1} \geq x_{N-1}^{*}$ we know that players 1 to $N-1$ have reached his threshold . By strategic substitutes, for all $i = i = 2, ..., N-1$ the component, of the extreme profile $\underline{s}_{-i}^{t-1}$, associated with the first player is $s_1(\cdot; \underline{k}_1)$. Then if player i, upon receiving a signal x_i and assuming that his opponents are using the strategy profile $\underline{s}_{-i}^{t-1}$, chooses action 1 he will choose action 1 for all $s_{-i} \in \widetilde{S}_{-i}^{t-1}$.

On the hand we need to prove that player N will increment the set of signal where he picks action 0, then if this player chooses action 0 when other players' strategy is $\overline{s}'^{t-1}_{-N}$, they will pick action 0 for all $s_{-1} \in \widetilde{S}_{-i}^{t}$. We now prove that the component, of the extreme profile $\overline{s}'^{t-1}_{-N}$, associated with the first player is $s_1(\cdot; \underline{k}_1)$ as well. Suppose by way of contradiction that the strategy is different from $s_1(\cdot; \underline{k}_1)$ for some $x_i \geq x_{N-1}^{*}$. Then $\exists$ a neighborhood $O \subseteq [x_i - \sigma, x_i + \sigma]$ such that on O $s_1(x_1) = 0$. Since for signals greater or equal than x_{N-1}^{*} we cannot have the $N-1$ dimensional profile $(0, ..., 0)$ on O. By anonymity permute player 1 with some player j $\underline{s}'^{t-1}_{j}(x_j) = 1$, then by strategic substitutes this permuted strategies higher payoff $\Delta\Pi_i$, and so $\underline{s}'^{t-1}_{-i}$ could not have been a maximizer of $\Delta\Pi_i$ on $\widetilde{S}_{-i}^{t-1}$.

Now, we conclude that for all signal greater or equal than x_{N-1}^{*} the extremal profiles $\underline{s}_{-i}^{t-1}$ $\forall i \in \{2, ..., N-1\}$and $\overline{s}'^{t-1}_{-N}$ coincide at least in their first component; both profiles consider player 1 playing the switching strategy $s_1(\cdot; \underline{k}_1)$. Then the subset of players $\{2..., N\}$ face a reduced game $G_{N-1}(\sigma, s_1 = 0)$. It easy to check that for the subset of signal $[x_{N-1}^{*}, \underline{X}+\sigma,]$ the reduced game $G_{N-1}(\sigma, s_1 = 0)$ satisfies assumptions A1 to A5. By the induction assumption we know that Lemma 5 holds for games with $N-1$ players, then for players 2 to N the round of elimination t coincides with the same round of the reduced game. More formally: for every player in $G_{N-1}(\sigma, s_1 = 0)$ there exist $\widehat{\underline{x}}_i^t$ and $\widehat{S}_i^t$, hence $\underline{x}_i^t = \widehat{\underline{x}}_i^t$ and $S_i^t = \widehat{S}_i^t$, i.e. S_i^t have the same structure than S_i^{t-1} for all $i \in \{2, ..., N\}$.

By assumption A5 (Players Order) and using the intermediate value theorem it is easy to check that there exist $\underline{x}_1^t < \underline{x}_2^t$ such that $\Delta\Pi_N(\overline{s}_{-1}^{t-1}, \underline{x}_1^t) = 0$. Therefore $S_1^t = \{s_1 : s_1(x_1) = 0 \; if \; x_1 < \underline{k}_1 \; and \; s_1(x_1) = 1 \; if \; x_1 \in (\underline{k}_1, \underline{x}_1^t) \cup (\overline{k}_1, \overline{X} + \sigma)\}$.

Proof of part ii) By the induction argument we know that in both reduced games, $G_{N-1}(\sigma, s_N = 0)$ and $G_{N-1}(\sigma, s_1 = 0)$ described above, the higher player completes the elimination of conditionally dominated strategies in a finite number of steps. Since we treated game $G_N(\sigma)$ as it were "composed" of two reduced games, player N in $G_N(\sigma)$ also completes the deletion process in a finite number of steps, i.e. $\exists \, t' \; s.t. \; \forall \, t \geq t', \; \underline{x}_N^t = \overline{k}_N$.■

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